DISCONTINUOUS SOLUTIONS IN PROBLEMS OF NONLINEAR HEAT CONDUCTION

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We propose a method of constructing discontinuous solutions for nonlinear problems in the theory of heat conduction. The method is a modification of the pivot methods for the solution of linear boundary-value problems. As an illustration we present numerical solutions of two problems.

1. Let us examine the following linear boundary-value problem:

$$c_1 y'' + c_2 y' + c_3 y + c_4 = 0,\ c_i = c_i(x),\ x \in (a,\ b); \tag{1}$$

$$(\alpha_0 y + \beta_0 y')|_{x=a} = \gamma_0,\ (\alpha_1 y + \beta_1 y')|_{x=b} = \gamma_1. \tag{2}$$

The solution of this problem can be constructed by the pivot method [1, 2]. Having approximated (1) by a system of finite-difference relations

$$c_1^k \frac{y_{k-1} - 2y_k + y_{k+1}}{h^2} + c_2^k \frac{y_{k+1} - y_{k-1}}{2h} + c_3^k y_k + c_4^k = 0, \tag{3}$$

where $c_i^k = c_i(x_k)$, $y_k = y(x_k)$, $x_k = a + kh$, $1 \le k \le n-1$, we can derive the formulas for the pivot coefficients

$$\begin{gathered} a_1^k \left[2c_1^k - a_1^{k-1} \left(c_1^k - \frac{h}{2} c_2^k \right) - h^2 c_3^k \right] = c_1^k + \frac{h}{2} c_2^k, \\ a_2^k \left[2c_1^k - a_1^{k-1} \left(c_1^k - \frac{h}{2} c_2^k \right) - h^2 c_3^k \right] = a_2^{k-1} \left(c_1^k - \frac{h}{2} c_2^k \right) + h^2 c_4^k. \end{gathered} \tag{4}$$

The formula for the reverse pivot

$$y_k = a_1^k y_{k+1} + a_2^k \tag{5}$$

with the coefficients a_1^k and a_2^k, which can be calculated according to (4), yields a solution for the boundary-value problem (1) and (2) that is correct to o(h).

In solving the boundary-value problems by the pivot method (3)-(5) the following conditions are significant: first of all, the solution of (1) is assumed to be continuous-differentiable on $(a,$ b); secondly, the calculation of the pivot coefficient in accordance with the recurrent formulas (4) or the analogous formulas (see [2]) assumes knowledge of the first coefficients a_1^0 and a_2^0 which are found from the left-hand boundary condition. In the general case this can be done only when at least one of the boundary conditions is linear. In section 2 we construct a modification of the pivot method by means of which it becomes possible to remove these limitations.

2. We have to find a solution for (1) to satisfy boundary conditions of the general form

$$\varphi_a(y_0,\ y_0',\ y_n,\ y_n') = 0,\ \varphi_b(y_0,\ y_0',\ y_n,\ y_n') = 0. \tag{6}$$

Moreover, with a finite number of points $\xi_p \in (a,$ b) the solutions and its derivative may experience discontinuities which are described by the following linear conditions:

$$\begin{gathered} a_{11}^p y_k^- + a_{12}^p \dot{y}_k^- = b_{11}^p y_k^+ + b_{12}^p \dot{y}_k^+ + c_1^p, \\ a_{21}^p y_k^- + a_{22}^p \dot{y}_k^- = b_{21}^p y_k^+ + b_{22}^p \dot{y}_k^+ + c_2^p,\ p = 1,\ 2,\ 3, \ldots,\ m, \end{gathered} \tag{7}$$

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where $y_k^{\mp} = y(x_k \mp 0)$, $\dot{y}_k^{\mp} = y'(x_k \mp 0)$.

We note that at the point ξ_p Eq. (1) loses significance and, of course, cannot be satisfied; the coefficients $c_i(x)$ at the points may therefore also experience discontinuities. Replacing $\dot{y}_k$ by the finite-difference expressions

$$\dot{y}_k^- = \frac{y_{k-2} - 4y_{k-1} + 3y_k^-}{2h} + o(h),$$

$$\dot{y}_k^+ = \frac{-y_k^+ + 4y_{k+1} - 3y_{k+2}}{2h} + o(h)$$

and using formulas (5) of the reverse pivot for the point $x_k = \xi_p + h$, we can bring (7) to the form

$$A_{11}^p y_k^- + A_{12}^p y_k^+ = B_1^p y_{k+2} + D_1^p,$$
$$A_{21}^p y_k^- + A_{22}^p y_k^+ = B_2^p y_{k+2} + D_2^p.$$

It can be demonstrated that in all cases of physical significance the determinant of this system does not vanish identically with respect to the step into which the segments $[\xi_p, \xi_{p+1}]$ of the interval (a, b) have been separated. With solution of the system we derive the formulas of reverse pivot for the point $x_k = \xi_p$

$$y_k^{\mp} = a_{1p}^{\mp} y_{k+2} + a_{2p}^{\mp}, \quad p = 1, 2, \ldots, m. \tag{8}$$

Unlike (4), the coefficients $a_{1p}^{\mp}$, $a_{2p}^{\mp}$ are functions not only of the pivot coefficients that have been calculated, but also of a_1^{k+1}, a_2^{k+2}. The latter are found from (8), (5), and (4), as well as from Eq. (3) for $x_k = \xi_p + h_{p+1}$. The recurrence formulas derived in this manner for the pivot coefficients, in conjunction with (8), make it possible to pass through the points ξ_p of the solution discontinuities described by (7) without resorting to iterations.

Let us now turn to the boundary conditions (6). Having introduced the notations $y_0' = \alpha$, $y_n = \beta$ and assuming α to be known, we find the expressions for the first pivot coefficients:

$$a_1^0 = 1 + \frac{h^2}{2} \frac{c_3^1}{c_1^1 + hc_2^1}, \quad a_2^0 = \frac{h}{2} \frac{hc_4^1 - 2\alpha\left(c_1^1 + \frac{h}{2} c_2^1\right)}{c_1^1 + hc_2^1}.$$

Having substituted these expressions into (4), we note that all a_1^k are not functions of α, while a_2^k are linear functions of

$$a_2^k = \gamma^k \alpha + \delta^k, \quad x_k \neq \xi_p. \tag{9}$$

It is easy to demonstrate that these properties are also present in the coefficients $a_{1p}^{\mp}$, $a_{2p}^{\mp}$.

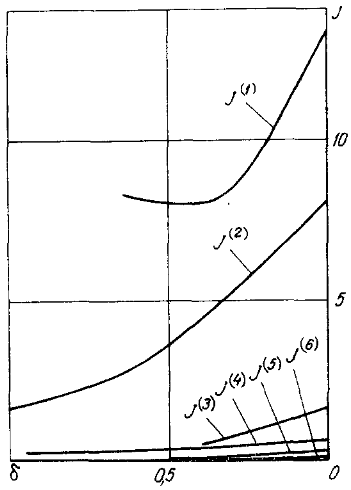

Fig. 1

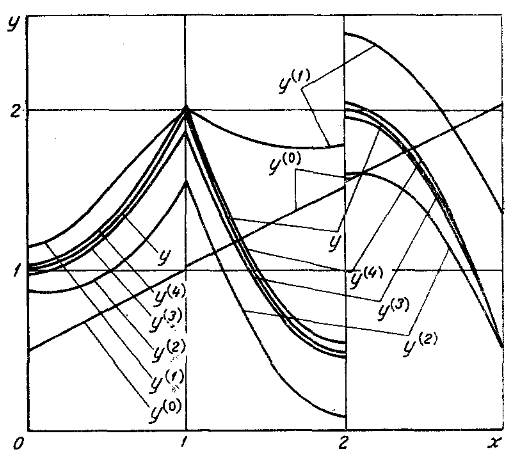

Fig. 2

Fig. 1. The functional of integral nonadjustment as a function of the parameter δ and the iteration number.

Fig. 2. Integral curves $y^k(x)$ as a function of the iteration number.

Hence it follows that the solution of the boundary-value problem (1), (6), and (7) can be presented in the form

$$y_k = \varkappa_1^k \alpha + \varkappa_2^k \beta + \varkappa_3^k, \; k = 0, 1, 2, \ldots, n. \tag{10}$$

Formula (10) is a generalization of the reverse-pivot formula for the case of discontinuous solutions satisfying the boundary conditions of the general form, i.e., Eq. (6). The coefficients $\varkappa_i^k$ are exclusive functions of the coefficients in Eq. (1) and the discontinuity conditions (7), and they can be calculated from the recurrent formulas:

$x_k \neq \xi_p$:

$$\varkappa_1^{k-1} = a_1^{k-1}\varkappa_1^k + \gamma^{k-1}, \quad \varkappa_2^{k-1} = a_1^{k-1}\varkappa_2^k, \quad \varkappa_3^{k-1} = a_1^{k-1}\varkappa_3^k + \delta^{k-1};$$

$x_k = \xi_p$:

$$\varkappa_{1p}^{\mp} = a_{1p}^{\mp}\varkappa_1^{k+2} + \gamma_p^{\mp}, \quad \varkappa_{2p}^{\mp} = a_{1p}^{\mp}\varkappa_2^{k+2}, \quad \varkappa_{3p}^{\mp} = a_{1p}^{\mp}\varkappa_3^{k+2} + \delta_p^{\mp};$$

$$\varkappa_1^n = 0, \; \varkappa_2^n = 1, \; \varkappa_3^n = 0.$$

We will now choose the parameters α and β so that the following system of equations (see (6) and (10)) is satisfied:

$$\begin{aligned} &\varphi_a\left(\varkappa_1^0\alpha + \varkappa_2^0\beta + \varkappa_3^0, \alpha, \beta, \frac{\sigma_1\alpha + \sigma_2\beta + \sigma_3}{2h}\right) = 0, \\ &\varphi_b\left(\varkappa_1^0\alpha + \varkappa_2^0\beta + \varkappa_3^0, \alpha, \beta, \frac{\sigma_1\alpha + \sigma_2\beta + \sigma_3}{2h}\right) = 0. \end{aligned} \tag{11}$$

The function y(x), calculated from (10) for α and β, satisfying system (11), in terms of its construction is a solution of the boundary-value problem (1), (6), and (7). Let us summarize the results obtained here. The boundary-value problem for (1) in conjunction with (6) and (7) can be solved by some modification of the pivot method, and on the first pass we calculate the coefficients a_1^k, γ_k, and δ^k; in the second pass we calculate the coefficients $\varkappa_i^k$, while in the third pass – in the reverse-pivot formulas (10) – we find the solution for $y(x_k)$. We will demonstrate in section 3 how the constructed algorithm can be used to find the solutions for the quasi-linear equations.

3. Let us examine the boundary-value problem for the quasi-linear equation

$$L[y] \equiv c_1 y'' + c_2 y' + c_3 y + c_4 = 0, \; c_i = c_i(x, y, y') \tag{12}$$

under conditions (6) and (7). The solution of this problem can be constructed by means of some iteration process which uses the pivot operator P as the iteration operator. P is understood to refer to the operator for the construction of the solution of the boundary-value problem (1), (6), and (7):

$$y^1 = P[y^0],$$

where (1) is a result of the substitution of the function $y^0(x)$ into the coefficients of (12). The algorithm for the iteration process involves the following. Let us choose or calculate some s-th approximation of y^s to satisfy or not satisfy conditions (6) or (7). Let us examine the single-parameter family of functions

$$y^s(x, \delta) = y^s + \delta(P[y^s] - y^s), \; \delta \in (-\infty, \infty). \tag{13}$$

From (13) we will select that element $y^s(x, \delta^*)$ which yields the minimum for the functional

$$J = \left(\int_a^b (L[y])^2 dx\right)^{\frac{1}{2}}. \tag{14}$$

It is this element that is taken as the (s + 1)-th approximation. For the proximity criterion y^{s+1} for the solution we assume the quantity

$$\Delta_{s+1} = \sup_{x \in G} |L[y^{s+1}]|, \; G \equiv \{x \mid x \in (a, b), \; x \neq \xi_p\}.$$

We note that from the very construction of the process functional (14) is nonincreasing and is obviously bounded from below by the function of the iteration number s and by the argument $|\delta|$ at each iteration. The example considered below can serve as an illustration of the foregoing.

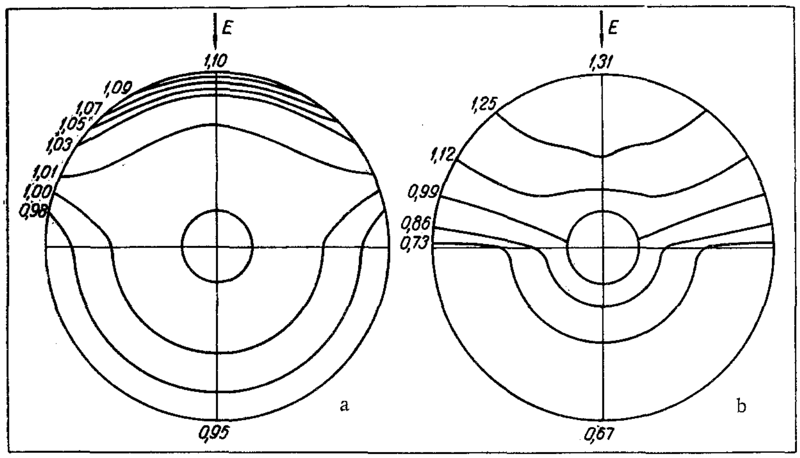

Fig. 3. Form of the isotherms in the cross section of a conic shell at the instant τ = 0.083 (the shell is markedly thickened) (a) and the steady-state distribution (b).

We are now called upon to solve the boundary-value problem for (12) with the coefficients

$$c_1 = x^2,\ c_2 = 0.5y',\ c_3 = 4(x^2 - y),\ c_4 = 4,\ x \in [0,\ 1];$$

$$c_1 = x^2,\ c_2 = -\frac{2}{3}(6 + y'),\ c_3 = x - \frac{1}{3}y',\ c_4 = -13,\ x \in (1,\ 2];$$

$$c_1 = x^2,\ c_2 = \frac{2}{3}(y' - 6),\ c_3 = x + \frac{1}{3}y',\ c_4 = 8,\ x \in (2,\ 3].$$

For the boundary conditions we assume

$$y_0' = 0,\ y_n = 0.5.$$

We will also require the solution to satisfy the following system of discontinuity conditions:

$$y(1-) = y(1+);\ y'(1-) = -y'(1+) - 1;$$

$$y(2-) = y(2+) - 1.5;\ y'(2-) = 0.$$

The problem is solved by the modified pivot method with iterations. Figures 1 and 2 show the curves $J^k_{(\delta)}$ and $y^k(x)$. As we can see from the figures, to find the solution – even when the initial approximation of y_0 has not been properly chosen – requires no more than 6 iterations.

The following function is an exact solution of the problem:

$$y = \begin{cases} x^2 + 1,\ x \in [0,\ 1]; \\ 1.5x^2 - 6x + 6.5,\ x \in (1,\ 2]; \\ -1.5x^2 + 6x - 4,\ x \in (2,\ 3]. \end{cases}$$

We note that the modified pivot method can be extended easily to the case in which we have to find the solution of the boundary-value problem for a system of quasi-linear equations of the form in (12).

4. Let us now turn to the equation of heat conduction. We will demonstrate how the above method can be applied to the solution of a mixed problem. For simplicity we will examine the one-dimensional case

$$u_t = c_1 u_{xx} + c_2 u_x + c_3,\ c_i = c_i(x,\ t,\ u,\ u_x). \tag{15}$$

The solution u(x, t) must satisfy at each instant of time $t > t_0$ Eq. (15), the initial condition $u(x, t_0) = \psi(x)$, as well as certain boundary conditions:

$$\varphi_a[t,\ u(a,\ t),\ u_x(a,\ t),\ u(b,\ t),\ u_x(b,\ t)] = 0,$$

$$\varphi_b[t,\ u(a,\ t),\ u_x(a,\ t),\ u(b,\ t),\ u_x(b,\ t)] = 0.$$

Moreover, if the thermophysical characteristics c, ρ, and k are not continuous functions of the variable x, as is the case, for example, in sandwich shells, the solution and its derivative $u_x(x, t)$ may experience discontinuities at certain points $\xi_p \in (a, b)$. If it is a steady-state problem that is being solved ($u_t \equiv 0$), without any changes the solution can be found in the manner described above. Here the arbitrary nature of the boundary conditions (6) makes it possible to provide for any mechanisms of heat transfer at the boundary of the region, and the general linear conditions of discontinuity (7) include both the special cases of the discontinuity conditions for ideal and real, passive (i.e., without heat sources) and active surface contacts.

The solution of the nonsteady problem can be found from the following implicit scheme:

$$c_1 u^j_{xx} + c_2 u^j_x + c_3 - \frac{u^j - u^{j-1}}{h_t} = 0, \qquad (16)$$

where $c_i = c_i(x, t^j, u^j, u^j_x)$.

If the solution $u^{j-1} \equiv u(x, t^j - h_t)$ is known, $u^j(x)$ is found by solving the boundary-value problem for (16), corresponding to the boundary conditions and the conditions of discontinuity. This can be accomplished in the manner described above. As is well known, the adopted implicit scheme is absolutely stable. This makes it possible to integrate (15) with a variable step h_t, i.e., to satisfy any prespecified accuracy.

As an example, let us examine a plane problem of establishing the equilibrium temperature distribution in a thin shell of conic shape.

The dimensionless equation has the form

$$u_\tau = A(u_{\xi\xi} + eu_\xi + du_{\varphi\varphi}),$$

where

$$e = \frac{m\cos\alpha}{1 + m\xi\cos\alpha}, \quad d = \frac{m^2}{(1 + m\xi\cos\alpha)^2}. \qquad (17)$$

The initial distribution was assumed to be uniform: $u(\xi, \varphi, \tau_0) \equiv 1$. The shell is subjected to external radiation of constant intensity E; on the inside wall, in addition to the radiation interaction with the shield, we make provision for convection heat transfer. The boundary conditions thus have the form

$$u_\xi(1, \varphi, \tau) = B\bar{E}\cos\alpha\,\overline{\cos\varphi} - Cu^4,$$

$$u_\xi(0, \varphi, \tau) = \alpha_c(u - u_i) + \beta_e(u^4 - u_s^4),$$

where $\overline{\cos\varphi} = 0.5[1 + \sin(\cos\varphi)]\cos\varphi$.

Equation (17) was replaced by a system of ordinary differential equations

$$A\left(u^j_{\xi\xi} + eu^j_\xi + d\,\frac{u^{j-1} - 2u^j + u^{j+1}}{h^2_\varphi}\right) - \frac{\partial u^j}{\partial\tau} = 0,\ j = 0, 1, \ldots, N_\varphi,\ u^{-1} \equiv u^{N_\varphi - 1},$$

and the boundary-value problem for this system was solved in accordance with scheme (16) by the modified pivot method. Figure 3 shows the form of the isotherms in one of the cross sections of the shell for two instants of time. The curves in Fig. 3a pertain to $\tau = 0.083$, while those in Fig. 3b relate to the steady-state temperature distribution.

NOTATION

ξ	is the dimensionless space variable, reckoned along the outside normal to the shell surface;
φ	is the angle of rotation about the axis of the cone;
A	is the dimensionless thermal diffusivity;
m	is a dimensionless parameter characterizing the thickness of the shell, and it is equal to the ratio of the physical thickness to the characteristic dimension of the cone;
α	is the cone angle;
B and C	are, respectively, the dimensionless absorptivity and emittance of the outside shell surface;
$\bar{E}$	is the dimensionless intensity of the incident radiation;
α_c	is the dimensionless coefficient of convection heat transfer;
u_i	is the dimensionless constant temperature of the gas filling the shell;
β_e	is the reduced emissivity of the interval between the inside wall and the screen;
u_s	is the dimensionless constant temperature of the screen near the inside shell surface.

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